Gradient like Morse-Smale dynamical systems on 4-manifolds

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Abstract. The complete invariant for gradient like Morse-Smale dynamical systems (vector fields and diffeomorphisms) on closed 4-manifolds are constructed. It is same as Kirby diagram in a case of polar vector field without fixed points of index 3.

1. Introduction.

The smooth dynamical system (vector field or diffeomorphism) is a Morse-Smale dynamical system if:

1) It has finite number of critical elements (periodic trajectories for diffeomorphisms, fixed points and closed orbits in the case of a vector field) and all of them are non degenerated (hyperbolic);
2) The stable and unstable manifolds of critical elements have transversal intersections;
3) Limit set of each trajectory is the critical element.

In papers [1] - [4] the topological classification of Morse-Smale dynamical systems on 2-manifolds and in [4] - [7] on 3-manifolds are obtained. Geteroclinic trajectories of the diffeomorphism are the trajectories laying in the intersection of stable and unstable manifolds of critical elements of the same indexes. Morse-Smale dynamic system is a gradient like if it hasn't contain closed trajectories in the case of a vector field [8] and geteroclinic trajectories in the case of diffeomorphism [9].

Two vector fields are topological equivalent, if there is a homeomorphism of manifold to itself, which maps integral trajectories into integral trajectories, keeping their orientations. This homeomorphism we shall call as conjugated. Two diffeomorphism f, $g : M \to M$ is topological conjugated, if there is a homeomorphism $h : M \to M$ such, that $hf = gh$.

The purpose of this paper is to obtain topological classification of gradient like Morse-Smale dynamical systems on 4-manifolds.

In section 2, we prove the criterion of topological equivalence of vector fields on 4-manifolds. We construct invariant of field that is 3-manifolds N with imbedded 2-spheres, framed links and surfaces. In section 3 and 4, we use it for construction the diagram of vector fields and prove the criteria of vector field equivalence using these diagrams. In section 5, we construct the diagram of diffeomorphism. In section 6, we investigate when the diagram can be realized as diagram of dynamical system.

2. Criterion of topological equivalence of vector fields.

Let M be a closed 4-manifold, X and X' are Morse-Smale vector fields.

Let $a_1, ..., a_k$ are fixed points of an index 0 of the field X, and $a'_1, ..., a'_k$ of the field X';

of the fixed points of index 0 and 1. We consider such tubular neighborhood $U(K)$ of this union, which does not contain other fixed points and such, that each trajectory has no more than one point of intersection with it. By $N = \partial U(K)$ we denote the boundary of this neighborhood for the field X and by N' for the field X'. Then these boundaries are 3-manifold.

We denote the stable and unstable manifolds of the fixed point X by $v(x)$ and $u(x)$. Let S_i are surfaces, which are intersections of unstable manifolds of the fixed points of an index 1 with manifold N. Then S_i is a set of not crossed spheres on manifold N, the supplement to which in N is three-dimensional spheres with cut out disks.

If $c_1, .., c_m$ are fixed points of an index 2, then the intersections $v_i = v(c_i) \cap N$ form a set of the closed curve on manifold N. Similarly for a field X' on manifold N' there is a set of the imbedded spheres and closed curve.

A handle of index k(k-handle) $h^k = D^k \times D^{n-k}$ attached to manifold M with boundary is the union $M \cup_f h^k$ along the boundary of M according to an embedding

$$f : S^{k-1} \times D^{n-k} \to \partial M.$$

$D^k \times \{0\}$ and $\{0\} \times D^{n-k}$ are called by core and cocore of the handle. $S^{k-1} \times \{0\}$ and $\{0\} \times S^{n-k-1}$ are called by attached and belt spheres of the handle h^k. Handles decompositions is a sequence of imbeddings

$$M_0 \subset M_1 \subset ... \subset M_N = M$$

such that M_0 is a union of n-disks (0-handles), M_{i+1} is obtained from M_i by gluing handle. Handles decompositions are isomorphic, if there exist a homeomorphism between the manifolds which maps the handles on the handles, the cores to the cores and the cocores to the cocores.

In another way the manifold $N = \partial U(K)$ together with the imbedded spheres and closed curves can be obtained if to consider the handle decomposition that is associated with the given vector field. This is such handle decomposition, in which to each handle there corresponds equally one fixed point (and on the contrary), and the core of each handle lay on the stable manifold of the appropriate fixed point, and the cocore lay on unstable one. Then $U(K)$ can be considered as the union of the handles of the index 0 and 1, the imbedded spheres as the belt sphere of the handles of the index 1, and closed curve as the attached spheres of the handles of the index 2. The gluing of each 2-handle is defined by a solid torus imbedded (neighborhood of the closed curve) in the manifold N. It can be given by a closed curve and a parallel on torus or simply by an integer (in a case if manifold N is three-dimensional sphere). This parallel is called by a framing.

We consider the intersections of the stable manifolds of the fixed points of an index 3 with manifold N. These intersections are surfaces spanned on the closed curves, that is such surfaces with boundaries, which interiors are imbedded in N, and the boundary component coincide with the closed curves (one closed curves can be contained in several component of the boundary). Thus the closed curve so much time coincides with boundary components of a surface, as many trajectories are in the intersections of unstable and stable manifolds of the appropriate fixed points of an index 2 and 3.

Theorem 1. *Vector field X is the topological equivalent to field X' if and only if there is a homeomorphism of manifolds $f : N \to N'$, which maps the spheres in the spheres, the closed curves in the closed curves, keeping their framing, and the surfaces in the surfaces.*

Proof. Necessity. Let φ is a conjugated homeomorphism between fields X and X'. We set homeomorphism f as follows. For each point $x \in N$ we shall consider a trajectory $g(x)$ which contain it. We define an image

$$f(x) = \varphi(\gamma(x)) \cap N'.$$

As the unstable manifolds of the fixed points of index 1 map onto unstable manifolds of the fixed points of an index 1 by homeomorphism φ, then the spheres map onto the spheres by homeomorphism f. Similarly, the closed curves map onto the closed curves, and the surfaces onto the surfaces. It is follows from existence of the homeomorphism φ that appropriate framings of the closed curve are equal.

Sufficiency. Let there is a homeomorphism $f : N \to N'$. We consider disks D_i, which lay on unstable integrated manifolds $u(b_i)$, contain points b_i and are limited by the spheres S_i. Then we can continue homeomorphisms from boundaries S_i of these disks up to homeomorphisms of disks that map integrated trajectories onto integrated trajectories (as each integrated trajectory, which has the intersections with a disk D_i, except the fixed points, must intersect the boundary of the disk). Then the disks D_i are decompose a neighborhood $U(K)$ on 4-disks D^4_j, each of which contain one fixed point of index 0.

Let us continue homeomorphisms from boundaries of these disks to an interior. For it we consider three-dimensional spheres S^3_j, which are the boundaries of such tubular neighborhoods of the fixed points of index 0, that each trajectory from an interior D^4_j crosses sphere S^3_j in a unique point. On each arch of the trajectory, which cross D^4_j we introduce a new parameterization in such a way that a point on S^3_j corresponds to value of the parameter $t = 0$, the point on the boundary D^4_j to $t = 1$ and the fixed point of the index 0 to $t = -1$. We demand that on each of the intervals (-1; 0) and (0; 1) the parameters t is proportional to the length of the arch in the Riemann metrics that is fixed for all trajectories. The homeomorphism of the boundaries of the disks D^4_j sets a correspondence between the trajectories inside this disk. A required homeomorphism maps a point of each integrated trajectory to a point of the appropriate integrated trajectory with the same parameter t.

For each closed curve v_i we shall consider its tubular neighborhood $U(v_i)$. Let W_i is such neighborhood of the appropriate fixed point c_i, which does not contain other fixed points and such, that boundary $\partial W_i = U(v_i) \cup V_i$ is a union of two solid torus. Thus the trajectories that have the intersection with $W_i \backslash c_i$ go in W_i through solid torus $U(v_i))$ and leave it through solid torus V_i.

We cut solid torus $U(v_i)$ from manifold N and glue solid torus V_i to obtain torus $\partial U(v_i) = \partial V$. Such operation we call by spherical surgery along v_i. Fulfilling the spherical surgeries along all v_i, we denote received manifolds by L. Thus between the solid torus $U(v_i)$ and V_i without middle circles (closed curve v_i and w_i on manifolds N and L) there is a homeomorphism, which maps a point $x \in N \backslash v_i$ to $\gamma(x) \in L$. Then homeomorphism of solid torus $U(v_i)$ and $U(v_i')$ induce a homeomorphism between $V_i \backslash w_i$ and $V_i' \backslash w_i'$. Using equality of framing of the closed curve v_i and v_i' we can extend this homeomorphism up to homeomorphism of the solid torus V_i and V_i'. We actually have constructed a homeomorphism between manifolds L and L'. Thus of surfaces from N map onto imbedded 2-sphere in L, and 2-spheres map onto surfaces. If homeomorphism from N to N' maps surfaces onto surfaces then homeomorphism from L to L' maps 2-sphere onto 2-spheres.

We extend the homeomorphism from boundaries W_i to their interior, and from manifold

N to $U(K)$. Constructed homeomorphism of manifold M will be required.

3. Diagram of a polar vector field.

Let vector field is polar (i.e. with one source and one sink). In this case, K is a bouquet of circles and manifold N is a connected sum $\#_n S^1 \times S^2$. We cut manifold N by 2-spheres. The obtained manifold is a 3-sphere without disks. Thus the closed curves are divided to arches, and surfaces to surfaces with these arches and arch in 2-spheres as its boundaries.

So constructed three-dimensional sphere, together with the imbedded pairs of 2-spheres, with given homeomorphisms of spheres of one pair, arches and closed curves with framing and surfaces we call by the diagram of a vector field. Two diagrams are equivalent if there is a homeomorphism of three-dimensional sphere of one diagram on three-dimensional sphere of other diagram, which maps

1) pairs of 2-spheres in pairs of 2-spheres and commute with given homeomorphisms of these spheres,
2) arch and closed curves in arches and closed curves and keep framing,
3) surfaces in surfaces.

If instead of surfaces we give the number of the fixed points (or handles) of index 3, we obtain the Kirby diagram of manifold M[10].

Proposition 1. *Two polar gradient like Morse-Smale vector fields are topological equivalent if and only if their diagrams are equivalent.*

We will prove proposition 1 in general case.

4. Diagram of a vector field in a general case.

As above, we construct manifold N with imbedded 2-dimensional spheres, framing circles and surfaces with boundary on this circles. Cutting manifold N by spheres, we obtain 3-dimensional spheres with the imbedded pairs of 2-dimensional spheres, as the boundaries of deleted disks, arches, which connect them, and closed curves with the framing and surfaces with the boundaries on this arch and closed curves.

The three-dimensional spheres, which is constructed in such way, together with the pairs of embedded 2-dimensional spheres, with given homeomorphisms of spheres of one pair, arches and closed curve with framing and surfaces with boundaries on its we call by the diagram of the vector field. Two diagrams are equivalent, if there is a homeomorphism of three-dimensional spheres of one diagram onto three-dimensional spheres of other diagram, which maps pairs of 2-dimensional spheres onto pairs of spheres, commuting with given homeomorphisms of these spheres, arches and closed curves onto arches and closed curves, keeping framing, and surfaces onto surfaces.

Proposition 2. *Two gradient like Morse-Smale vector fields are topological equivalent if and only if their diagrams are equivalent.*

Proof. Necessity. Required homeomorphism of three-dimensional spheres can be obtained in result of restriction of the homeomorphism $f : N \to N'$ on the correspondent three-dimensional spheres with deleted disks and extension of it on these disks.

Sufficiency. If there exist a homeomorphism between three-dimensional spheres, we consider its restriction on these spheres with deleted three-dimensional disks, which are

with given homeomorphisms of 2-dimensional spheres, we can glue it in homeomorphism of three-dimensional manifolds $f : N \to N'$. Using the theorem 1 we shall receive necessary conditions of the proposition.

5. Topological conjugation of diffeomorphisms.

Let $f : M \to M$ is a gradient like Morse-Smale diffeomorphism. Similarly to item 2 and 3 we shall construct the diagram of this diffeomorphism. Then action of diffeomorphism f on integrated manifolds of the periodic saddle points induce the map of such sets:

1) Three-dimensional spheres,
2) Pairs of 2-dimensional spheres, which is embedded in them,
3) Arches and circles with framing,
4) Surfaces,
5) Spheres with holes that obtained by cutting the manifold L on 2-dimensional spheres, which are the images of surfaces with spherical surgery.

These maps we call by internal.

Theorem 2. *Two gradient like Morse-Smale diffeomorphisms f and g are toplogicaly cojugated if and only if there exists a isomorphism of their diagrams, which sets equivalence between them and commute with internal maps.*

Proof. The necessity of conditions follows from construction. Let's show sufficiency. As it was done in the theorem 1, we construct homeomorphism h between manifolds, which maps stable manifolds onto stable and unstable onto unstable. Thus

$$h(f(U)) = g(h(U)),$$

where U is a part of stable or unstable manifold, on which it is decomposed by other manifolds. Similarly, how it was done for 3-dimensional manifolds [6,7], this homeomorphism can be corrected up to required conjugated homeomorphism.

6. Realization of dynamical system with given invariant.

In this section we describe when the three-dimensional spheres, together with the pairs of imbedded spheres, with given homeomorphisms of spheres of one pair, arches and closed curves with framing and surfaces with boundary on its are the diagram of some vector field.

We start with necessary conditions on surfaces. Since after spherical surgery along the curves and arches with framing surfaces should became spheres, then they are homeomorphic to spheres with deleted disks. Their intersections with torus, which are the boundary of the curve or arches with framing, consist of circles that set framing. The diagrams that satisfy to these conditions, we call admissible.

1) We shall consider the diagrams, in which there are no pairs of imbedded spheres and surfaces. Each such diagram is the Kirby diagram. If it is the diagram of a vector field, then this field have one source and sink and haven't the fixed points of an index 1 and 3. Then after spherical surgery along the framing link given by this diagram, the three-dimensional sphere should turn out. Therefore Kirby diagram will be the diagram of a vector field in only case when, when it is the diagram of three-dimensional sphere. From [10] it is follows, that it will be in only case when from this diagram it is possible

from the diagram of trivial knot with framing $+1$ or -1. Fenn and Rourke proved, that these two movements can be replaced by one — blow-up and its reverse — blow-down [11].

Nevertheless, these criteria do not give an algorithm, which check if the Kirby diagram is the diagram of three-dimensional sphere or not. Such methods of recognition of three-dimensional sphere having the Heegaard diagram are in [12] and [13]. Using Rolfsen idea Christian Mercat show how the Heegaard diagrams can be constructed from the Kirby diagrams.

2) We show, as from any admissible diagram it is possible to obtain the Kirby diagram (without embedded 2-dimensional spheres). In the beginning we get rid from superfluous 1 and 3 handles (that is pairs of embedded spheres and surfaces), which can be reduced together with additional 0 and 4 handles. If the diagram consists of several three-dimensional spheres, then using connectedness of the manifold Ì we can find the pair of 2-dimensional spheres (1-handle) laying in different three-dimensional spheres. If we take thee connected sum of three-dimensional spheres along this pair, we obtain the diagram, with one three-dimensional sphere fewer than starting diagram. We repeat this procedure until there will be one three-dimensional sphere.

Similarly it is possible to get rid from superfluous surfaces, consistently deleting them. Thus at each stage the surface will be superfluous, if there not exists the simple closed curve, which has transversal intersections with given surfaces and does not have with others.

We replace the stayed pairs of spheres and surfaces with 2-handles. Thus we shall receive the diagram from 1).

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